On CFTs with Four-Fermion Interactions

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Talk at the Nambu Memorial Conference

THE UNIVERSITY OF CHICAGO

March 12, 2016

The Nambu-Jona-Lasinio Model

PHYSICAL REVIEW VOLUME 122, NUMBER 1 APRIL 1, 1961

Dynamical Model of Elementary Particles Based on an Analogy with Superconductivity. I*

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(Received October 27, 1960)

It is suggested that the nucleon mass arises largely as a self-energy of some primary fermion field through the same mechanism as the appearance of energy gap in the theory of superconductivity. The idea can be put into a mathematical formulation utilizing a generalized Hartree-Fock approximation which regards real nucleons as quasi-particle excitations. We consider a simplified model of nonlinear four-fermion interaction which allows a γ_5-gauge group. An interesting consequence of the symmetry is that there arise automatically pseudoscalar zero-mass bound states of nucleon-antinucleon pair which may be regarded as an idealized pion. In addition, massive bound states of nucleon number zero and two are predicted in a simple approximation.

The theory contains two parameters which can be explicitly related to observed nucleon mass and the pion-nucleon coupling constant. Some paradoxical aspects of the theory in connection with the γ_5 transformation are discussed in detail.

- The NJL 4-fermion interaction

$$L=-\bar{\psi}\gamma_\mu\partial_\mu\psi+g_0[(\bar{\psi}\psi)^2-(\bar{\psi}\gamma_5\psi)^2].$$

has a U(1) chiral symmetry under

$$\psi\rightarrow\exp[i\alpha\gamma_5]\psi,\quad \bar{\psi}\rightarrow\bar{\psi}\exp[i\alpha\gamma_5],$$

- Key ideas: dynamical breaking of the U(1) chiral symmetry and appearance of a Nambu-Goldstone boson.
- We want to discuss four-fermi interactions as a function of dimensionality, d, and the number of fermion components, N.
- Will consider the closely related Gross-Neveu Model:

$$S_{\rm GN} = -\int d^d x \left(\bar{\psi}_i \gamma^\mu \partial_\mu \psi^i + \frac{g}{2} (\bar{\psi}_i \psi^i)^2 \right) \qquad i = 1, 2, \ldots \tilde{N}$$

- The NJL interaction in 4 dimensions may also be written as $-\frac{1}{2} g_0 [(\bar{\Psi}\gamma_\mu \Psi)^2 - (\bar{\Psi}\gamma_\mu \gamma_5 \Psi)^2]$.

Thirring Model and QED

- The first term is the Thirring interaction.
- We will discuss the multi-flavor version of the Thirring model.
- For 2<d<4, and at least for large N, it has a UV fixed point which is equivalent to the IR fixed point of QED:

$$S = \int d^d x \left(\frac{1}{4e^2} F^{\mu\nu} F_{\mu\nu} - \sum_{i=1}^{N_f} \bar{\psi}_i \gamma^\mu (\partial_\mu + i A_\mu) \psi^i \right)$$

- The induced effective action for the gauge field at the fixed point is

$$\int d^d x d^d y \left(\frac{1}{2} A^\mu(x) A^\nu(y) \langle J_\mu(x) J_\nu(y) \rangle_0 + \mathcal{O}(A^3) \right) \qquad J_\mu = \bar{\psi}_i \gamma_\mu \psi^i$$

The Gross-Neveu Model

$$S_{\mathrm{GN}} = -\int d^d x \left(\bar{\psi}_i \gamma^\mu \partial_\mu \psi^i + \frac{g}{2} (\bar{\psi}_i \psi^i)^2 \right)$$

- In 2 dimensions it has some similarities with the 4-dimensional QCD.
- It is asymptotically free and exhibits dynamical mass generation.
- Another 2-dimensional model with similar physics is the O(N) non-linear sigma model.
- In dimensions slightly above 2 both the O(N) and GN models have weakly coupled UV fixed points.

2+ ε expansion

- The beta function and the critical value of the coupling are

$$\beta = \epsilon g - (N-2)\frac{g^2}{2\pi} + (N-2)\frac{g^3}{4\pi^2} + (N-2)(N-7)\frac{g^4}{32\pi^3} + \mathcal{O}(g^5)$$

$$g_* = \frac{2\pi}{N-2}\epsilon + \frac{2\pi}{(N-2)^2}\epsilon^2 + \frac{(N+1)\pi}{2(N-2)^3}\epsilon^3 + \mathcal{O}(\epsilon^4)\,,$$

- $N = \tilde{N}\mathrm{Tr}\mathbf{1}$ is the number of components of the Dirac fermions.
- The 2+ε expansion for scaling dimensions of simplest operators, like the fermion or fermion bilinear, have been developed. See an excellent review by Moshe and Zinn-Justin.
- Similar expansions in the O(N) sigma model. Brezin, Zinn-Justin

4-ε expansion

- The O(N) sigma model is in the same universality class as the Wilson-Fisher O(N) model:

$$S = \int d^d x \left(\frac{1}{2} (\partial \phi^i)^2 + \frac{\lambda}{4} (\phi^i \phi^i)^2 \right)$$

- It has a weakly coupled IR fixed point in 4-ε dimensions.
- Using the two ε expansions, the conformal field theories with different N may be studied in the range 2<d<4. This is a great practical tool for CFTs in d=3.

The Gross-Neveu-Yukawa Model

- The GN model is in the same universality class as the GNY model Zinn-Justin; Hasenfratz, Hasenfratz, Jansen, Kuti, Shen

$$S_{\rm GNY} = \int d^d x \left(-\bar{\psi}_i (\not{\partial} + g_1 \sigma)\psi^i + \frac{1}{2}(\partial_\mu \sigma)^2 + \frac{g_2}{24}\sigma^4 \right)$$

- Has an IR fixed point in 4-ε dimensions

$$g_{1\star} = \sqrt{\frac{16\pi^2 \epsilon}{N+6}}\,,$$

$$g_{2\star} = 16\pi^2 \epsilon \frac{24N}{(N+6)\left((N-6) + \sqrt{N^2+132N+36}\right)}$$

- Using the two ε expansions, we can study the Gross-Neveu CFT in the range 2<d<4.

QED$_d$ in the 4-ε expansion

$$S = \int d^dx \left(\frac{1}{4e^2} F^{\mu\nu} F_{\mu\nu} - \sum_{i=1}^{N_f} \bar{\psi}_i \gamma^\mu (\partial_\mu + i A_\mu) \psi^i \right)$$

- Use dimensional continuation such that ψ^i are 4-component spinors in d<4.
- The usual QED in 4d with N_f massless Dirac fermions is connected to QED$_3$ with $2N_f$ two-component fermions.
- A weakly coupled IR fixed point in d=4-ε

$$\beta = -\frac{\epsilon}{2} e + \frac{4N_f}{3} \frac{e^3}{(4\pi)^2} + \mathcal{O}(e^5)$$

$$e_*^2 = 6\epsilon\pi^2 / N_f + \mathcal{O}(\epsilon^2)$$

Higher Spin AdS/CFT

- When N is large, the conformal O(N), GN and QED models have an infinite number of higher spin currents whose anomalous dimensions are of order 1/N.
- Their singlet sectors have been conjectured to be dual to the Vasiliev interacting higher-spin theories in d+1 dimensional AdS space.
- One passes from the dual of the free to that of the interacting large N theory by changing boundary conditions at AdS infinity. IK, Polyakov; Leigh, Petkou; Sezgin, Sundel; for an excellent review, see Giombi, Yin.

1/N Expansion

- Both the GN and the scalar O(N) models have "double-trace" interactions

$$S_\lambda = S_{\mathrm{CFT}_0} + \lambda \int d^d x O(x)^2$$

$$\langle O(x)O(y)\rangle_0 = \frac{C_O}{|x-y|^{2\Delta_O}} = C_O \frac{(4\pi)^{d/2}\Gamma\left(d/2-\Delta_O\right)}{4^{\Delta_O}\Gamma\left(\Delta_O\right)} \int \frac{d^d p}{(2\pi)^d} e^{ip(x-y)} (p^2)^{\Delta_O - d/2}$$

- Use the Hubbard-Stratonovich transformation

$$S_\lambda = S_{\mathrm{CFT}_0} + \int d^d x \sigma O - \frac{1}{4\lambda} \int d^d x \sigma^2$$

- Induced quadratic term for the auxiliary field

$$\begin{aligned} S[\sigma] &= -\frac{1}{2} \int d^d x d^d y \; \sigma(x)\sigma(y)\langle O(x)O(y)\rangle_0 - \frac{1}{4\lambda} \int d^d x \sigma^2 \\ &= -\frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \sigma(p)\sigma(-p) \left(C_O \frac{(4\pi)^{d/2}\Gamma\left(d/2-\Delta_O\right)}{4^{\Delta_O}\Gamma\left(\Delta_O\right)} (p^2)^{\Delta_O - d/2} + \frac{1}{2\lambda} \right) \end{aligned}$$

- At the critical point the induced propagator is

$$G_\sigma(p) = \langle \sigma(p)\sigma(-p)\rangle = -\frac{4^{\Delta_O}\Gamma\left(\Delta_O\right)}{C_O(4\pi)^{d/2}\Gamma\left(d/2-\Delta_O\right)}(p^2)^{d/2-\Delta_O} \equiv \tilde{C}_\sigma(p^2)^{d/2-\Delta_O}$$

$$G_\sigma(x,y) = \frac{(d/2-\Delta_O)\sin\left((d/2-\Delta_O)\pi\right)\Gamma\left(d-\Delta_O\right)\Gamma\left(\Delta_O\right)}{\pi^{d+1}C_O|x-y|^{2(d-\Delta_O)}} \equiv \frac{C_\sigma}{|x-y|^{2(d-\Delta_O)}}$$

- The 1/N expansion is found using this induced propagator. In the GN model, Gracey

$$\Delta_\psi = \frac{d}{2} - \frac{1}{2} + \eta^{\rm GN} \qquad\qquad \eta^{\rm GN} = \eta_1^{\rm GN}/N + \eta_2^{\rm GN}/N^2 + \mathcal{O}(1/N^3)$$

$$\eta_1^{\rm GN} = \frac{\Gamma(d-1)(d-2)^2}{4\Gamma(2-\frac{d}{2})\Gamma(\frac{d}{2}+1)\Gamma(\frac{d}{2})^2}$$

- This result agrees with the two ε expansions.

Nambu-Jona-Lasinio CFT

- Need two Hubbard-Stratonovich fields

$$L = i\bar{\psi}^i \partial\!\!\!/ \psi^i + \sigma\bar{\psi}^i\psi^i + i\pi\bar{\psi}^i\gamma^5\psi^i - \frac{1}{2g^2}(\sigma^2+\pi^2)$$

- Chiral U(1) global symmetry.
- The fermion anomalous dimension calculated long ago to order $1/N^2$ Gracey

$$\begin{aligned} \eta_1 &= -\frac{2\Gamma(2\mu-1)}{\Gamma(\mu-1)\Gamma(1-\mu)\Gamma(\mu+1)\Gamma(\mu)} \\ \eta_2 &= \eta_1^2\left[\Psi(\mu) + \frac{2}{\mu-1} + \frac{1}{2\mu}\right] \end{aligned} \qquad d = 2\mu$$

C_J and C_T

$$\langle J^a_\mu(x_1) J^b_\nu(x_2)\rangle = C_J \frac{I_{\mu\nu}(x_{12})}{(x_{12}^2)^{d-1}} \delta^{ab} \,,$$

$$\langle T_{\mu\nu}(x_1) T_{\lambda\rho}(x_2)\rangle = C_T \frac{I_{\mu\nu,\lambda\rho}(x_{12})}{(x_{12}^2)^{d}}$$

$$I_{\mu\nu}(x) \equiv \delta_{\mu\nu} - 2\frac{x_\mu x_\nu}{x^2} \,,$$

$$I_{\mu\nu,\lambda\rho}(x) \equiv \frac{1}{2}(I_{\mu\lambda}(x) I_{\nu\rho}(x) + I_{\mu\rho}(x) I_{\nu\lambda}(x)) - \frac{1}{d}\delta_{\mu\nu}\delta_{\lambda\rho}$$

- C_J determines the universal charge or spin conductivity.
- C_T enters in many contexts including the entanglement entropy. It is one of the natural measures of degrees of freedom in a CFT.
- In d=2 satisfies the Zamolodchikov C-theorem, but there are counterexamples in d>2.

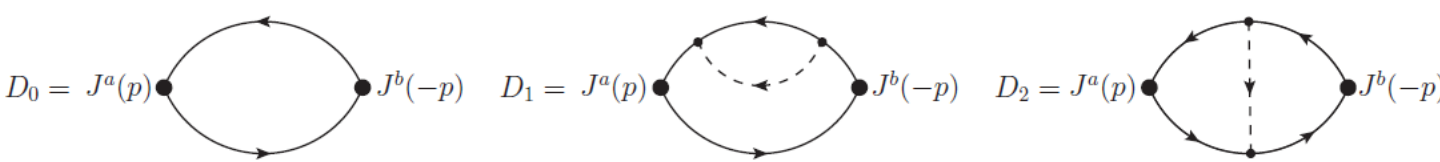
Diagrammatic Calculations

- In large N theories

$$C_J = C_{J0}\left(1 + \frac{C_{J1}}{N} + \frac{C_{J2}}{N^2} + \mathcal{O}(1/N^3)\right),$$
$$C_T = C_{T0}\left(1 + \frac{C_{T1}}{N} + \frac{C_{T2}}{N^2} + \mathcal{O}(1/N^3)\right)$$

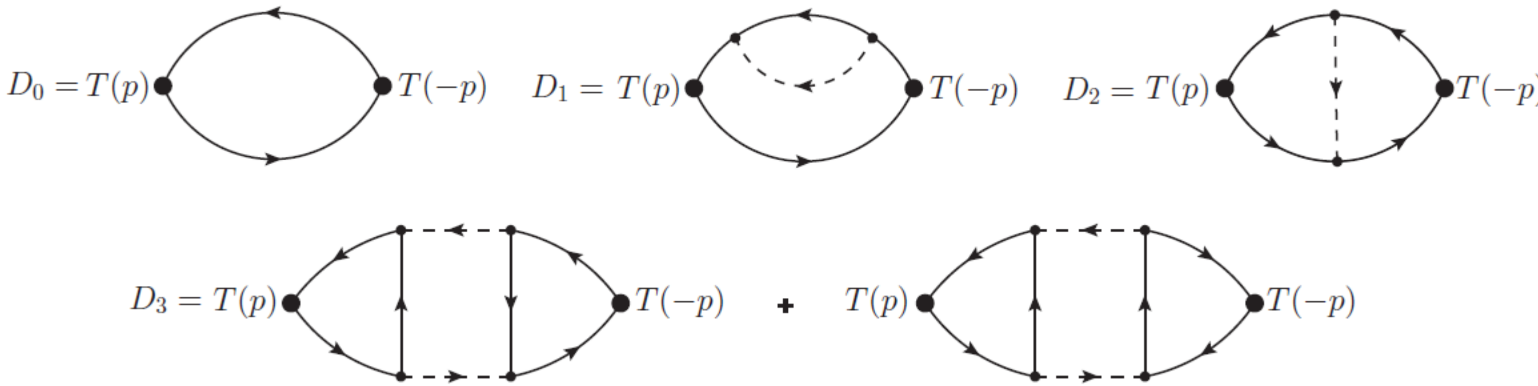
- In the GN and O(N) models the leading 1/N corrections were calculated as a function of d

Diab, Fei, Giombi, IK, Tarnopolsky



$$C_{J1}^{\mathrm{GN}} = -\frac{8(d-1)}{d(d-2)}\eta_1^{\mathrm{GN}} \qquad C_{J1}^{\mathrm{O(N)}} = -\frac{8(d-1)}{d(d-2)}\eta_1^{\mathrm{O(N)}}$$

- Calculation of C_{T1} is more subtle



$$C_{T1}^{\mathrm{GN}} = -4\eta_1^{\mathrm{GN}}\left(\frac{\mathcal{C}_{\mathrm{GN}}(d)}{d+2} + \frac{(d-2)}{d(d+2)(d-1)}\right)$$

$$\mathcal{C}_{\mathrm{GN}}(d) = \psi(2-\frac{d}{2}) + \psi(d-1) - \psi(1) - \psi(\frac{d}{2})$$

- Agrees with the $2+\varepsilon$ and $4-\varepsilon$ expansions, which can be developed for the GN model using standard renormalized perturbation theory.

O(N) model

- Using the diagrammatic techniques we find

$$C_{T1}^{\mathrm{O(N)}} = -2\left(\frac{2\mathcal{C}_{\mathrm{O(N)}}(d)}{d+2} + \frac{d^2+6d-8}{d\,(d^2-4)}\right)\eta_1^{\mathrm{O(N)}}$$

$$\eta_1^{\mathrm{O(N)}} = \frac{2\Gamma(d-2)\sin(\pi\frac{d}{2})}{\pi\Gamma(\frac{d}{2}-2)\Gamma(\frac{d}{2}+1)}$$

$$\mathcal{C}_{\mathrm{O(N)}}(d) = \psi(3-\frac{d}{2}) + \psi(d-1) - \psi(1) - \psi(\frac{d}{2})$$

- This agrees with Petkou's much earlier calculation based on the OPE and the knowledge of exact 4-point function.
- Has been tested in 4-ε, but the 2+ε expansion for the O(N) sigma model is subtle due to IR divergences.
- The formula for C_{T1} is not as universal as for C_{J1}.

Three Dimensions

- In the physically interesting dimension

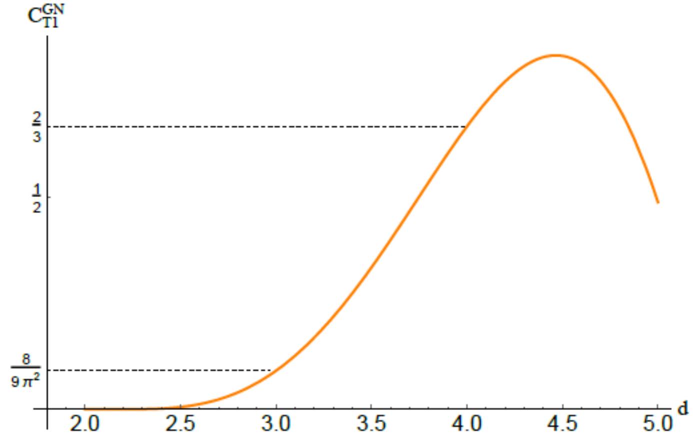
$$C_J^{\mathrm{GN}}|_{d=3} = C_{J0}^{\mathrm{GN}}\left(1 - \frac{64}{9\pi^2 N} + \mathcal{O}(1/N^2)\right)$$
$$C_T^{\mathrm{GN}}|_{d=3} = C_{T0}^{\mathrm{GN}}\left(1 + \frac{8}{9\pi^2 N} + \mathcal{O}(1/N^2)\right)$$

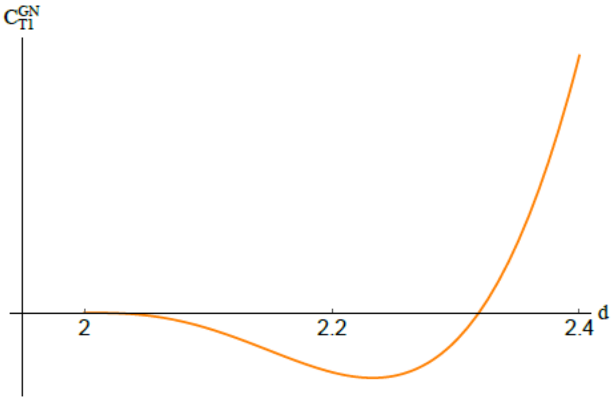
- For the O(N) model we instead have

$$C_J^{\mathrm{O(N)}}|_{d=3} = C_{J0}^{\mathrm{O(N)}}\left(1 - \frac{64}{9\pi^2 N} + \mathcal{O}(1/N^2)\right)$$
$$C_T^{\mathrm{O(N)}}|_{d=3} = C_{T0}^{\mathrm{O(N)}}\left(1 - \frac{40}{9\pi^2 N} + \mathcal{O}(1/N^2)\right)$$

- The 1/N correction to C_T is much smaller in the GN than in the O(N) model.

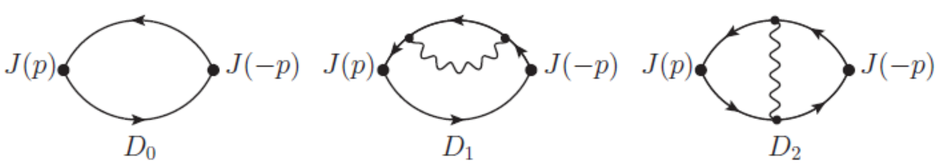
- Interestingly, the RG inequality $C_T^{\mathrm{UV}} > C_T^{\mathrm{IR}}$ applies to both models in d=3.
- However, it does not work in all dimensions between 2 and 4, as this plot of C_{T1} for the GN model shows.
- The C_T inequality fails for 2<d<2.3.





C_J and C_T in Conformal QED

- Here the 1/N corrections are calculated using the induced photon propagator.
- To find C_J we calculated Giombi, Tarnopolsky, IK



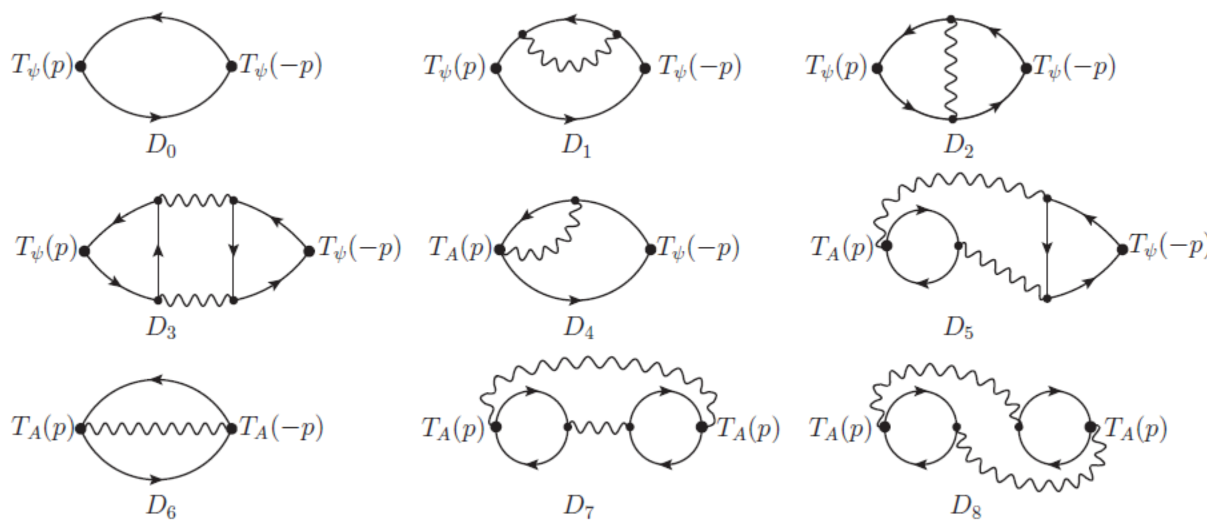
$$C_{J0} = \mathrm{Tr}\mathbf{1}\frac{1}{S_d^2}$$

$$C_{J1}(d) = \eta_{m1}\left(\frac{3d(d-2)}{8(d-1)}\Theta(d) + \frac{d-2}{d}\right) \qquad \Theta(d) \equiv \psi'(d/2) - \psi'(1)$$

- The electron mass anomalous dimension is

$$\eta_{m1}(d) = -\frac{2(d-1)\Gamma(d)}{\Gamma(\frac{d}{2})^2\Gamma(\frac{d}{2}+1)\Gamma(2-\frac{d}{2})}$$

- The calculation of C_T requires more diagrams because $T = T_\psi + T_A$ Huh and Strack



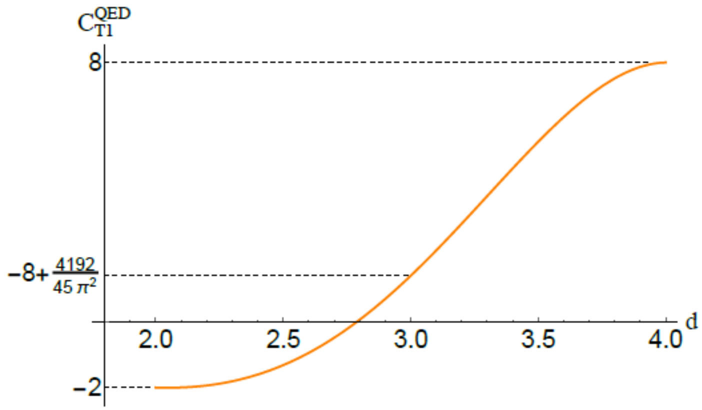
- With an analytic regulator we find

$$C_{T1}(d) = \eta_{m1}\left(\frac{3d(d-2)}{8(d-1)}\Theta(d) + \frac{d(d-2)}{(d-1)(d+2)}\Psi(d) - \frac{(d-2)(3d^2+3d-8)}{2(d-1)^2 d(d+2)}\right)$$

$$\Psi(d) \equiv \psi(d-1) + \psi(2-d/2) - \psi(1) - \psi(d/2-1)$$

- Agrees with the 4-ε expansion for QED.
- In d=2 agrees with the exact result for multi-flavor Schwinger model

$$C_T|_{d=2} = \frac{N}{S_2^2}\left(1 - \frac{2}{N}\right)$$



- In d=3 we find

$$C_{J1}(3) = \frac{736}{9\pi^2} - 8 \approx 0.285821$$
$$C_{T1}(3) = \frac{4192}{45\pi^2} - 8 \approx 1.43863$$

Sphere Free Energy in Continuous d

- A natural quantity to consider is Giombi, IK

$$\tilde{F} = \sin(\pi d/2) \log Z_{S^d} = -\sin(\pi d/2) F$$

- In odd d, this reduces to IK, Pufu, Safdi

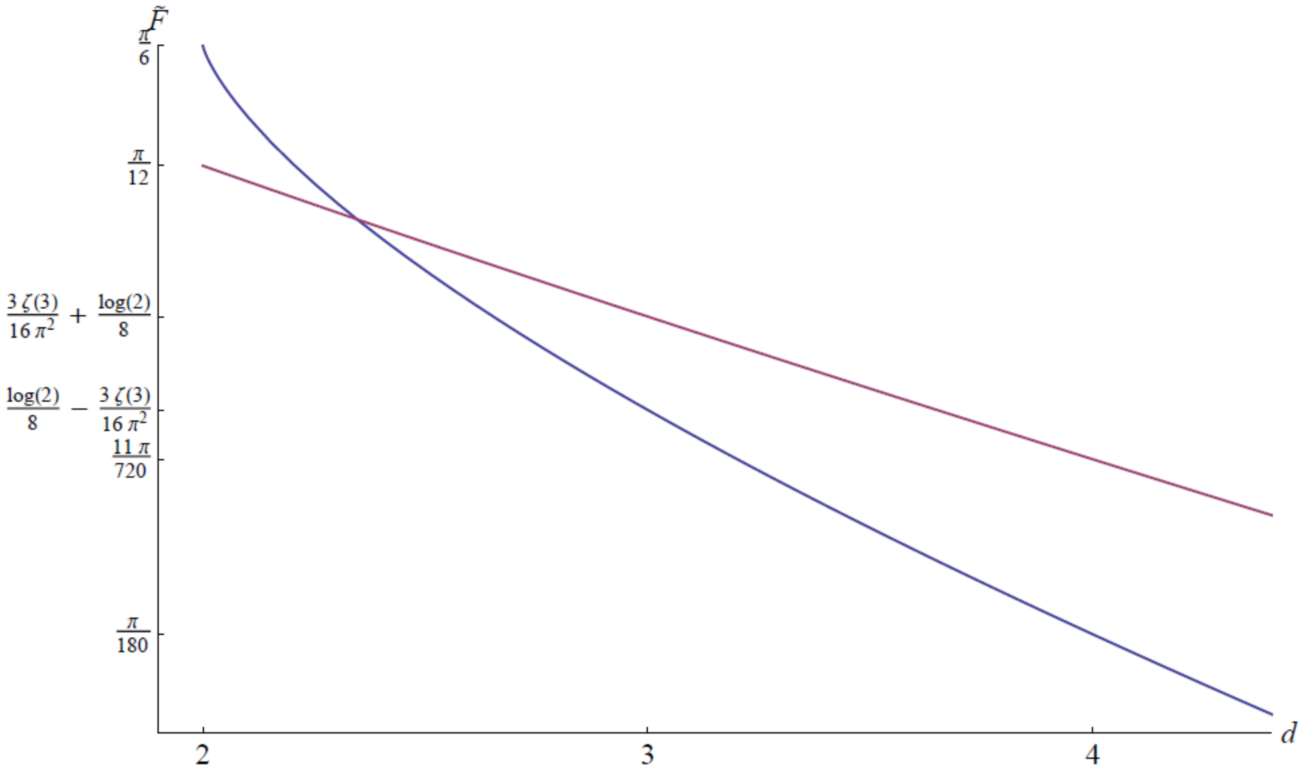
$$\tilde{F} = (-1)^{\frac{d+1}{2}} F = (-1)^{\frac{d-1}{2}} \log Z_{S^d}$$

- In even d, -log Z has a pole in dimensional regularization whose coefficient is the Weyl *a*-anomaly. The multiplication by $\sin(\pi d/2)$ removes it.
- $\tilde{F}$ smoothly interpolates between *a*-anomaly coefficients in even and ``F-values" in odd d.
- Gives the universal term in the entanglement entropy across d-2 dimensional sphere.

Free Conformal Scalar and Fermion

$$\tilde{F}_s = \frac{1}{\Gamma(1+d)} \int_0^1 du\, u \sin \pi u\, \Gamma\left(\frac{d}{2}+u\right) \Gamma\left(\frac{d}{2}-u\right) \ ,$$
$$\tilde{F}_f = \frac{1}{\Gamma(1+d)} \int_0^1 du\, \cos\left(\frac{\pi u}{2}\right) \Gamma\left(\frac{1+d+u}{2}\right) \Gamma\left(\frac{1+d-u}{2}\right)$$

- Smooth and positive for all d.



F-theorem in Continuous d?

- Based on the known F- and a-theorems, it is natural to ask whether

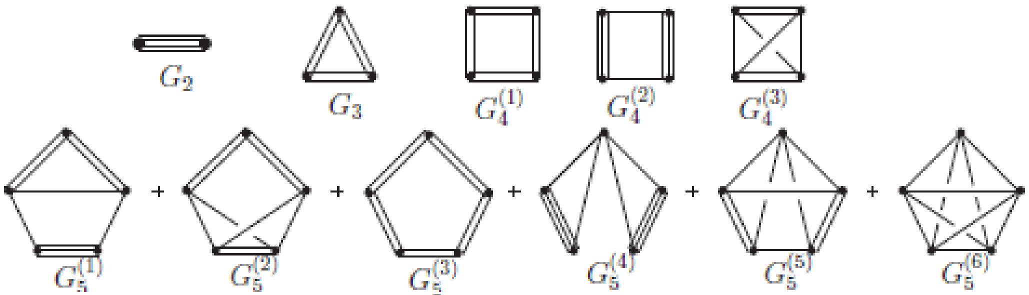
$$\tilde{F}_{UV} > \tilde{F}_{IR}$$

holds in continuous dimension *d*.

- We have calculated $\tilde{F}$ in various examples of CFTs that can be defined in continuous dimension, including double-trace flows in large N CFTs and perturbative O(N) and GN fixed points in the epsilon-expansion.
- In all unitary examples that we considered, we find that $\tilde{F}$ indeed decreases under RG flow.

O(N) Model in d=4-ε

- We performed a perturbative calculation of F to order λ^5
  Fei, Giombi, IK, Tarnopolsky



- The poles in the above diagrams fix the curvature beta functions to the needed order. At the IR fixed point, we get the final result for $\tilde{F} = -\sin(\pi d/2) F$:

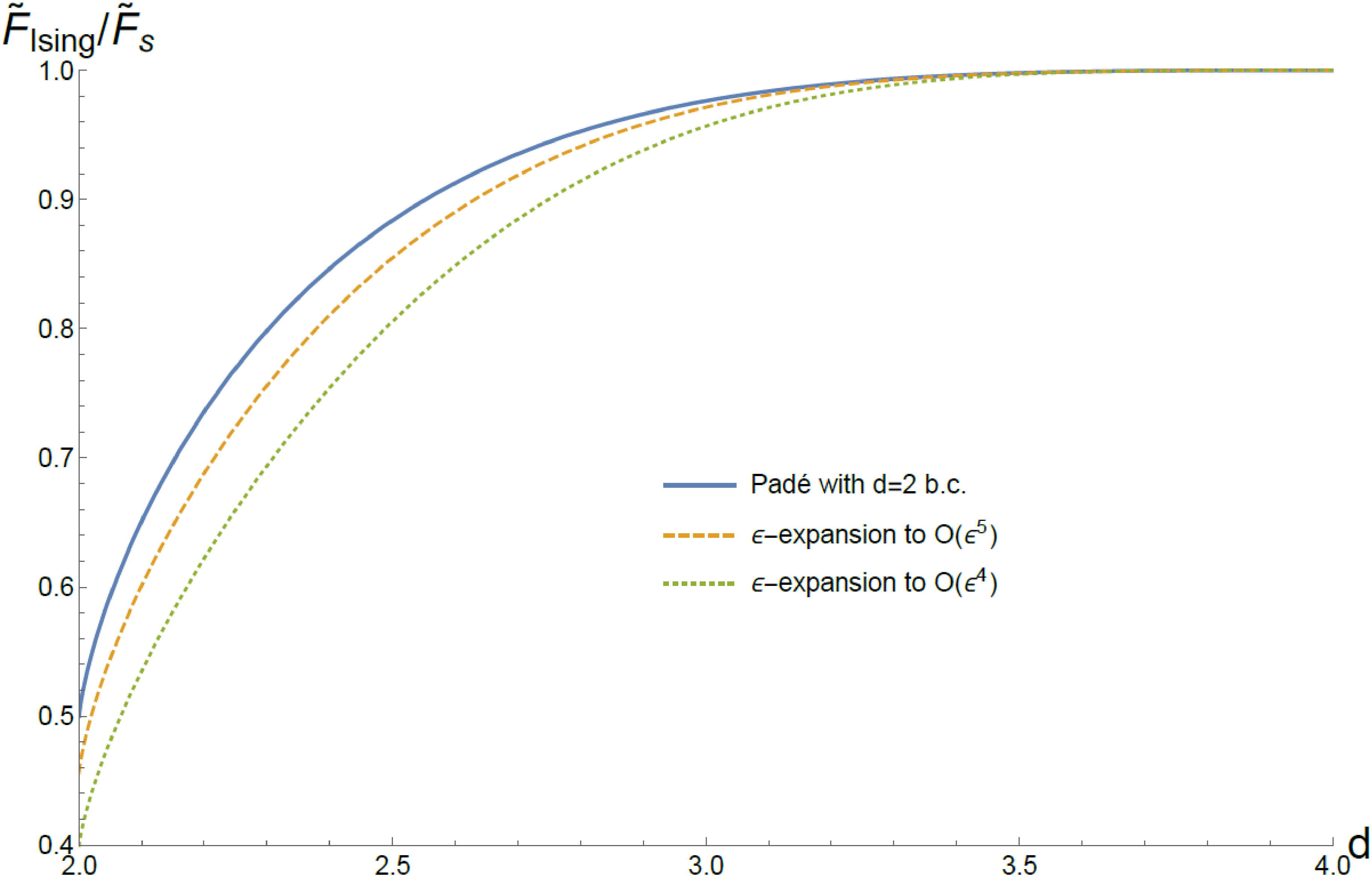
$$\begin{aligned}\tilde{F}_{\rm IR} =& N\tilde{F}_s(\epsilon) - \frac{\pi N(N+2)\epsilon^3}{576(N+8)^2} - \frac{\pi N(N+2)(13N^2+370N+1588)\epsilon^4}{6912(N+8)^4} \\ &+ \frac{\pi N(N+2)}{414720(N+8)^6}\Big(10368(N+8)(5N+22)\zeta(3) - 647N^4 - 32152N^3 \\ &\qquad -606576N^2 - 3939520N + 30\pi^2(N+8)^4 - 8451008\Big)\epsilon^5 + \mathcal{O}(\epsilon^6)\end{aligned}$$

F for the 3d Ising Model

- Extracting precise estimates from the ε–expansion requires resummation. A simple approach is to use Pade approximants

$$\text{Padé}_{[m,n]}(\epsilon) = \frac{A_0 + A_1\epsilon + A_2\epsilon^2 + \ldots + A_m\epsilon^m}{1 + B_1\epsilon + B_2\epsilon^2 + \ldots + B_n\epsilon^n}$$

- For the Ising model (N=1), we expect $\tilde{F}$ to be a smooth function of *d*, such that near *d*=4 it reproduces the perturbative ε–expansion we computed, and in *d*=2 it reproduces the exact central charge of the 2d Ising CFT: c=1/2.
- The accuracy of the Pade approximants can be improved if we impose the exact value c=1/2 (which in terms of $\tilde{F}$ corresponds to $\tilde{F} = \pi/12$) as a boundary condition at *d*=2



$$\tilde{F} = \tilde{F}_s + \tilde{F}_{\text{int}} = \frac{\pi}{180} + 0.0205991\epsilon + 0.0136429\epsilon^2 + 0.00670643\epsilon^3 + 0.00264883\epsilon^4 + 0.000927589\epsilon^5 + O(\epsilon^6)$$

$$\tilde{F}_s = \frac{\pi}{180} + 0.0205991\epsilon + 0.0136429\epsilon^2 + 0.00690843\epsilon^3 + 0.00305846\epsilon^4 + 0.0012722\epsilon^5 + O(\epsilon^6)$$

- Using the constrained Pade approximant method, we get

$$\frac{F_{\text{3d Ising}}}{F_{\text{free sc.}}} \approx 0.976$$

- Consistent with the F-theorem.
- The value of F for 3d Ising is very close to the free field value!
- A similar result was found for c_T in the conformal bootstrap approach *El-Showk et al.*

$$c_T^{\text{3d Ising}} / c_T^{\text{3d free scalar}} \approx 0.9466$$

- The dimension of ϕ is 0.518… which is only 3.6% above the free field value.

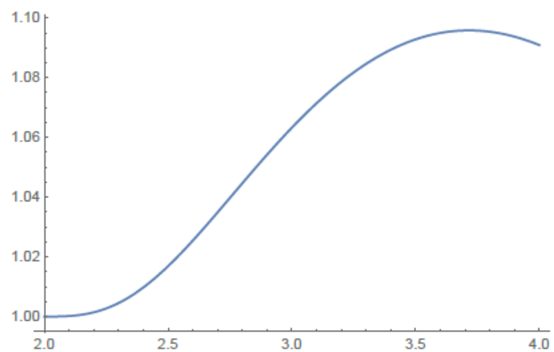
Sphere Free Energy for Gross-Neveu

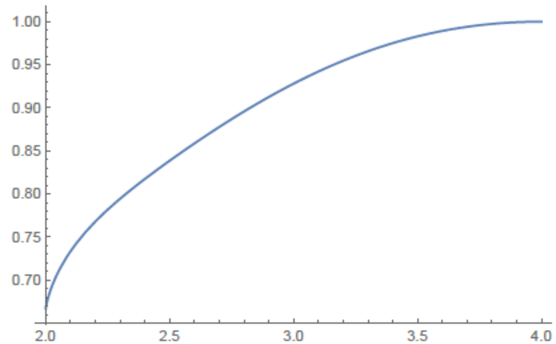
- Has been calculated using the 2+ε and 4-ε expansions. Fei, Giombi, IK, Tarnopolsky, to appear
- The 2+ε expansion is under good control; no IR divergences:

$$\tilde{F} = N\tilde{F}_f + \frac{N(N-1)\pi\epsilon^3}{48(N-2)^2} - \frac{N(N-1)(N-3)\pi\epsilon^4}{32(N-2)^3} + \mathcal{O}(\epsilon^5)$$

- The plot of ratios to $N\tilde{F}_f$ and to $N\tilde{F}_f + \tilde{F}_s$ for N=4
- Once again

$$\tilde{F}_{UV} > \tilde{F}_{IR}$$





QED_3

- Consider QED in d=3 with massless fermions

$$S = \int d^3x \left(\frac{1}{4e^2} F^{\mu\nu} F_{\mu\nu} - \sum_{i=1}^{N_f} \bar{\psi}_i \gamma^\mu (\partial_\mu + i A_\mu) \psi^i \right)$$

- Here ψ^i are N_f *4-component* spinors, and γ^μ are (three of) the usual 4x4 Dirac matrices.
- Writing $\psi^i = (\chi_1^i, \chi_2^i)$, the action can be written equivalently in terms of $2N_f$ two-component spinors χ_1^i, χ_2^i.
- The model has $SU(2N_f)$ global symmetry.
- This is often called a “chiral” symmetry. A parity invariant mass term $m\bar{\psi}_i \psi^i$ would explicitly break $SU(2N_f)$ down to $SU(N_f) \times SU(N_f) \times U(1)$.

QED$_3$ at large N$_f$

- At large N_f, the free energy on S^3 in the IR CFT can be computed in the $1/N_f$ expansion. The first non-trivial correction comes from the determinant of the induced kinetic operator for the gauge field IK, Pufu, Sachdev, Safdi

$$F_{\rm conf} = N_f \left(\frac{\log(2)}{2} + \frac{3\zeta(3)}{4\pi^2} \right) + \frac{1}{2}\log\left(\frac{\pi N_f}{4}\right) + \mathcal{O}(\frac{1}{N_f})$$

- In the UV limit, the free energy F_{UV} diverges due to the log(R) dependence of the Maxwell term
- Thus, there is no contradiction with F-theorem $F_{UV}>F_{IR}$, despite the fact that the $\log(N_f)$ term in F_{conf} can grow without bound for large N_f

QED_3 at finite N_f and symmetry breaking

- For sufficiently large N_f, this interacting fixed point is expected to be stable, because there are no relevant operators preserving $SU(2N_f)$ and parity.
- As we lower N_f, a widely discussed scenario is that for N_f less or equal than some critical value N_{crit}, the model displays spontaneous symmetry breaking according to the pattern

$$SU(2N_f) \to SU(N_f) \times SU(N_f) \times U(1)$$

- Expected to be due to the condensation of

$$\bar{\psi}\psi = (\bar{\chi}_1\chi_1 - \bar{\chi}_2\chi_2)$$

which breaks $SU(2N_f)$ but preserves parity Pisarski; Appelquist et al

- If SSB occurs, then the IR theory consists of the $2N_f^2$ Nambu-Goldstone bosons, plus an extra massless scalar (dual photon).

QED_3

- At $N_f = N_{crit}$ a quartic fermion operator (invariant under $SU(2N_f)$ and parity) can become relevant in the IR, and render the IR fixed point unstable towards the symmetry breaking Di Pietro et al; Kaveh, Herbut; Kaplan, Lee, Son, Stephanov; Braun et al
- We would like to use the F-theorem as a tool to provide some additional constraint on the value of N_{crit} (a similar F-theorem approach to N_{crit} was considered earlier by Grover)
- Since N_f is small, we use the epsilon expansion and the dimensional continuation of

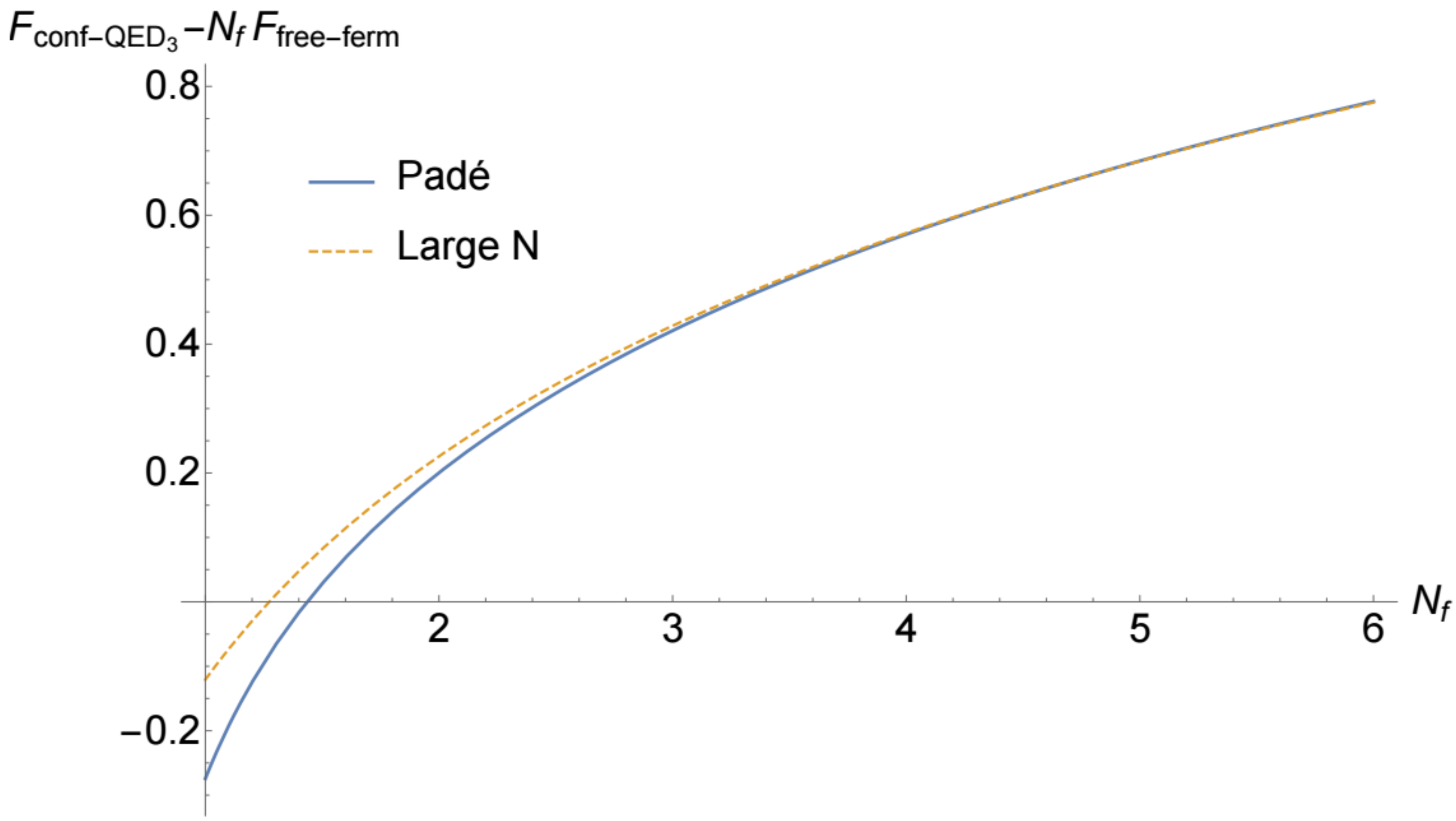
$$\tilde{F} = -\sin(\pi d/2) F$$

to estimate the value of F in QED_3

- At the IR fixed point

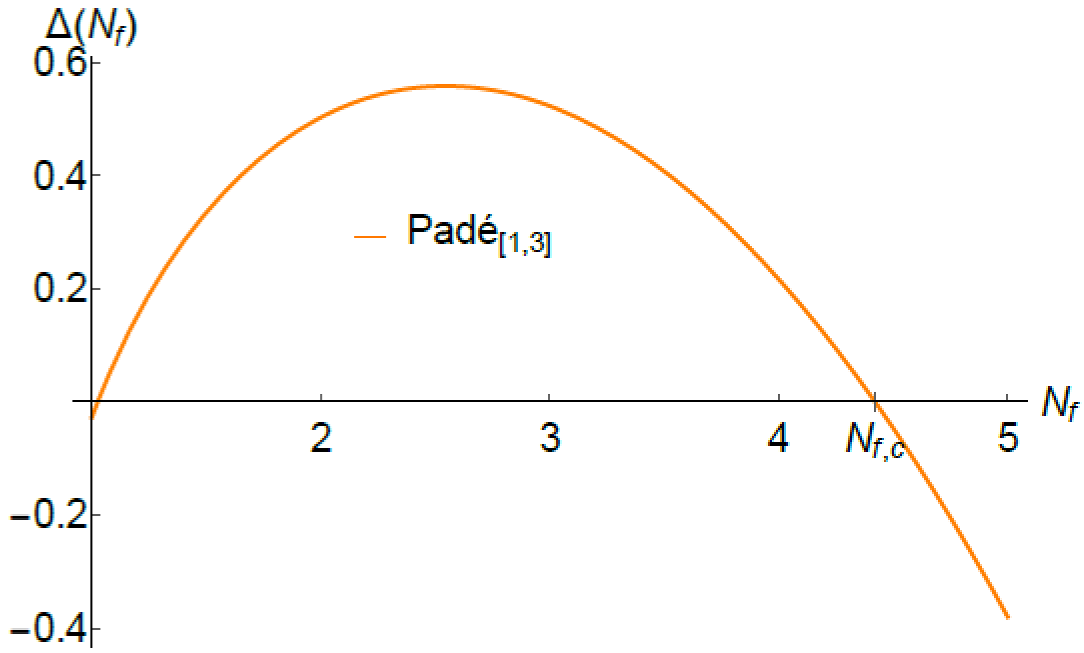
$$\tilde{F}_{\rm conf} = N_f \tilde{F}_{\text{free-ferm}} - \frac{1}{2}\sin(\frac{\pi d}{2})\log(\frac{N_f}{\epsilon}) + \frac{31\pi}{90} - 1.2597\epsilon - 0.6493\epsilon^2 + 0.8429\epsilon^3 + \frac{0.4418\epsilon^2}{N_f} - \frac{0.6203\epsilon^3}{N_f} - \frac{0.5522\epsilon^3}{N_f^2} + \mathcal{O}(\epsilon^4)$$

- The term $31\pi/90$ is from the a-anomaly coefficient of the d=4 Maxwell field.
- A new feature is the non-analytic term ~ $\log(N_f/\varepsilon)$. This originates from the free Maxwell contribution, which contains the term $\log(e^2R^{4-d})$
- The R dependence drops out at $e=e_*$, as a result of delicate cancellations between the free Maxwell term and terms due to interactions. Consistent with the expected conformal invariance in the IR.



- The plot of the Pade resummed ε-expansion evaluated at ε=1, compared to the d=3 large N_f expansion result shows that they are very close already at N_f ~ 3

F-theorem and N_{crit}



Plot of $\Delta(N_f) = F_{\text{conf}}(N_f) - F_{\text{SB}}(N_f)$, using Padé$_{[1,3]}$

- We conclude that QED_3 must be in the SU($2N_f$) invariant conformal phase for $N_f \geq 5$

Another Estimate for SB?

- The far UV theory of free fermions and decoupled Maxwell field is not conformal. Define C_T via the 2-point function of

$$T \equiv z^\mu z^\nu T_{\mu\nu} \qquad\qquad z^\mu z^\nu \delta_{\mu\nu} = 0$$

- Then $C_T^{\mathrm{UV}} = \frac{12N_f + 9}{32\pi^2}$
- For the SB phase with Nambu-Goldstone bosons $C_T^{\mathrm{IR}} = \frac{3(2N_f^2 + 1)}{32\pi^2}$
- IF we assume $C_T^{\mathrm{UV}} > C_T^{\mathrm{IR}}$ then

$$N_{f,\mathrm{crit}} = 1 + \sqrt{2} \approx 2.414$$

Above 4 Dimensions

- The O(N), GN and QED models make sense at least to all orders in the 1/N expansion.
- Interesting weak coupling expansions near even dimensions, where σ has a local action.
- For example, in 6-ε dimensions find the cubic O(N) symmetric theory

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi^i)^2 + \frac{1}{2}(\partial_\mu \sigma)^2 + \frac{g_1}{2}\sigma(\phi^i\phi^i) + \frac{g_2}{6}\sigma^3$$

- It has an IR fixed point for sufficiently large N. Results there agree with the 1/N corrections found for O(N) model as a function of d. Fei, Giombi, IK, Tarnopolsky; Gracey

Conclusions

- The ε and 1/N expansions in the O(N), Gross-Neveu, QED, and other vectorial models, are of much interest for applications to condensed matter physics and statistical mechanics.
- They provide "checks and balances" for the new numerical results using the conformal bootstrap.
- They serve as nice playgrounds for the RG inequalities (C-theorem, a-theorem, F-theorem) and for the higher spin AdS/CFT correspondence.
- The Nambu-Jona-Lasinio CFT is mostly work for the future.